Evidence for fractional topological charge in SU(2) pure Yang-Mills theory*

Robert G. Edwards, Urs M. Heller and Rajamani Narayanan [a]

[a] SCRI, The Florida State University, Tallahassee, FL 32306-4130, USA

We investigate the spectral flows of the hermitian Wilson-Dirac operator in the fundamental and adjoint representations on two ensembles of pure SU(2) gauge field configurations at the same physical volume. We find several background gauge field configurations where the index of the hermitian Wilson-Dirac operator in the adjoint representation is not four times the index in the fundamental representation. This could imply a topological basis for the existence of degenerate vacua in supersymmetric Yang-Mills theories.

The overlap formalism for constructing a chiral gauge theory on the lattice [1] provides a natural definition of the index, I, of the associated chiral Dirac operator. The index is equal to half the difference of negative and positive eigenvalues of the hermitian Wilson-Dirac operator

$$\mathrm{H}_L(m) = \begin{pmatrix} \mathrm{B}(U) - m & \mathrm{C}(U) \\ \mathrm{C}^\dagger(U) & -\mathrm{B}(U) + m \end{pmatrix} \tag{1}$$

– we will use an unconventional sign for the mass term throughout! – where C is the (naive) lattice transcription of the Weyl term, and B is the usual Wilson term (covariant Laplacian). In the naive continuum limit, $\mathrm{H}(m) = \gamma_5(\gamma_\mu D_\mu - m)$, m can be chosen as any positive value, since zero eigenvalues of $\mathrm{H}(m)$, and therefore eigenvalues crossing zero as function of m, are only possible at $m = 0$. Furthermore, crossings of zero are associated with the topology of the background gauge field: the topological charge, Q, is equal to the index of the chiral Dirac operator.

On the lattice, because of the additive mass renormalization, the crossings of zero occur at positive m and spread out in m. It is easy to see that no eigenvalues of $\mathrm{H}_L(m)$ can be zero for $m < 0$. Since in the free case the first doublers become massless at $m = 2$ we restrict ourselves to $m < 2$. A simple way to compute the index I is to compute the lowest eigenvalues of $\mathrm{H}_L(m)$ at some suitably small m before any crossings of zero occurred. Then m is slowly varied and the number and direction of zero crossings are tracked. The net number at some m_t is the index of the overlap chiral Dirac operator.

We have applied this procedure to compute the index, which we take as the definition of topological charge on the lattice, of various gauge field ensembles with gauge group SU(3) in [2]. We found that the zero crossings start occurring at some, ensemble dependent, $m_1 > 0$ and continue occurring for all m in $m_1 < m < 2$. We found a monotonic relation between the crossing point m and the size of the corresponding zero mode, with the crossings for larger objects occurring at smaller m. All crossings for objects with size larger than about two lattice spacings occurred within a small region of $m > m_1$. All later zero crossings correspond to small objects of size about one to two lattice spacings. These small objects do not seem to have physical effects and, for example, do not affect the topological susceptibility [2–4].

Here we consider two ensembles of SU(2) background gauge fields generated using the standard Wilson action with periodic boundary conditions on the gauge fields. One is at a lattice coupling of $\beta = 2.4$ on an 8^4 lattice and the other is at a lattice coupling of $\beta = 2.6$ on a 16^4 lattice. The couplings were chosen so that both lattices had the same physical volume ($a = 0.12$ fm at $\beta = 2.4$ and $a = 0.06$ fm at $\beta = 2.6$) and are in the confined phase ($\beta_c = 2.5115$ at $N_\tau = 8$ and $\beta_c = 2.74$ at $N_\tau = 16$) [5]. The topological susceptibility obtained from the index in the fun-

*Presented by U. M. Heller at *Lattice '98*.

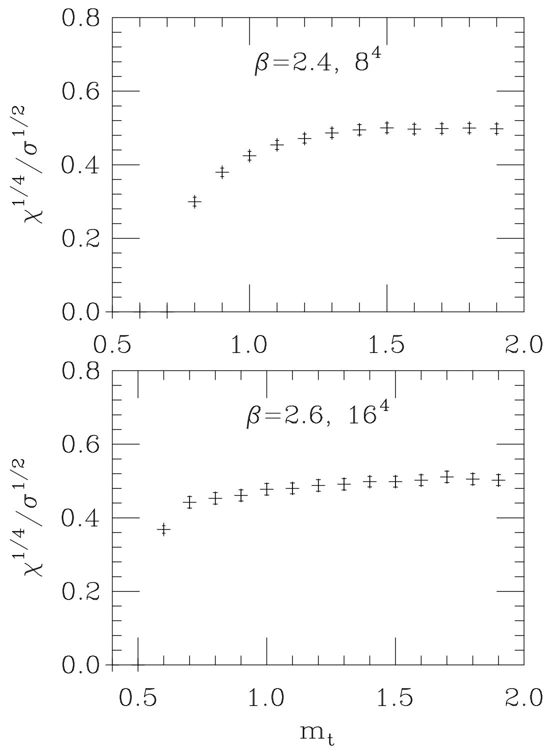

Figure 1. The topological susceptibility as function of the mass m_t used to define the index of the overlap chiral Dirac operator.

damental representation is shown in Fig. 1. As for SU(3) we find that the topological susceptibility becomes independent of the mass m_t used to define the index, once m_t is in the range of zero crossings associated with objects of small size.

The index of the massless Dirac operator in the adjoint representation of the SU(N) gauge group in a background field of topological charge Q is $I_a = 2NQ$ [6]. Classical instantons carry an integer topological charge and therefore give a non-zero expectation value to an operator that contains $2N$ Majorana fermions, $\lambda(x_1)\lambda(x_1)\cdots\lambda(x_N)\lambda(x_N)$, independent of the x_i (see [7] for a recent review). On very general grounds [8], we know that the theory has N degenerate vacua in a finite physical volume with a different non-zero expectation value for $\lambda(x)\lambda(x)$ in each of these vacua, and hence a $\lambda(x)\lambda(x)$ condensate in the infinite volume limit. Self-dual twisted gauge field configurations with a fractional topological charge of $\frac{1}{N}$ [9] can be used to explain these N degenerate vacua [10]. A non-perturbative computation of $\langle\lambda(x)\lambda(x)\rangle$ on the lattice would typically be done with conventional periodic boundary conditions for both the gauge fields and fermions in order to achieve a supersymmetric theory in the continuum limit. In this situation the picture of N degenerate vacua in a finite physical volume does not clearly emerge.

To shed more light on this question, we use the overlap definition of the index of the chiral Dirac operator in the adjoint representation, I_a, of gauge group SU(2) to study the topological content of gauge field configurations in Monte Carlo generated ensembles. Since the fermion is in the real representation of the gauge group, the spectrum of the hermitian Wilson Dirac operator, $\mathrm{H}_L(m)$, is doubly degenerate. Therefore, the index of the associated chiral Dirac operator can only be even valued. This is to be expected. Otherwise, we would have to explain away unphysical expectation values of an odd number of fermionic observables in an otherwise well defined theory. However, it is possible to obtain any even value for the index. The issue one has to address is if all possible even values are realized for the index in an ensemble of SU(2) gauge field backgrounds or if only multiples of four are observed. If only multiples of four are observed then one would conclude that the gauge field background behaves as if it were made up of classical instantons with small fluctuations, and we would not be able to explain the N degenerate vacua by topological means. On the other hand, if we observe any even value for the index that are not just multiples of four, the background gauge fields cannot be thought of as being made up of classical instantons and we would have evidence for N degenerate vacua arising out of a topological mechanism.

We considered fifty configurations, each, in the two SU(2) ensembles already used to compute the topological susceptibility from the index I_f in the fundamental representation, shown in Fig. 1 [11]. We have doubled this number by including the parity transformed partner of every configuration since this symmetrize's the distribution of the indices. The computation of the index of the over-

lap chiral Dirac operator in the adjoint representation was done exactly as for the fundamental representation. We found configurations for which $I_a = 4I_f$, *i.e.,* where the topological charge as given by the fundamental and adjoint indices agrees. The corresponding zero modes were compatible with representing the same topological objects. But we found also many configurations in both ensembles that do not satisfy the relation $I_a = 4I_f$, and we found several configurations where I_a is not a multiple of four. The occurrence of a significant number of configurations with values of I_a that are not multiples of four is taken as evidence for the existence of gauge field configurations with fractional topological charge in our ensemble since $Q = \frac{I_a}{4}$ is the continuum relation between the topological charge and the index of the Dirac operator in the adjoint representation.

Having provided some evidence for the existence of fractional topological charge on the lattice at finite lattice spacing, we now address the question of whether these are pure lattice artifacts. For this purpose, we define the quantity $\Delta = I_a - 4I_f$ for each configuration in both ensembles. Note that Δ takes on only even values. The probability of finding a certain value of Δ, $p(\Delta)$, is plotted for the two ensembles in Fig. 2. We find that $p(\Delta)$ for $|\Delta| > 2$ decreases as one goes toward the continuum limit at a fixed physical volume. However, $p(\pm 2)$ does not decrease indicating that it might remain finite in the continuum limit.

In summary we have presented some preliminary evidence for fractional topological charge on the lattice. By studying two ensembles with different lattice spacings we have argued that there is a reasonable indication that this is not a lattice artifact. If this result survives the continuum limit it provides a topological basis for N degenerate vacua in supersymmetric SU(N) gauge theories.

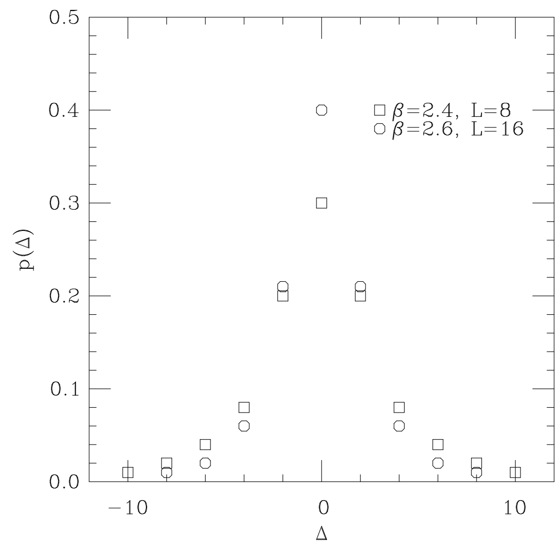

Figure 2. $p(\Delta)$ versus Δ for the two ensembles where $\Delta = I_a - 4I_f$.

Acknowledgements

This research was supported by DOE contracts DE-FG05-85ER250000 and DE-FG05-96ER40979. Computations were performed on the QCDSP and CM-2 at SCRI.

REFERENCES

1. R. Narayanan and H. Neuberger, *Nucl. Phys.* **B443** (1995) 305; *Phys. Rev. Lett.* **71** (1993) 3251.
2. R.G. Edwards, U.M. Heller, R. Narayanan and R.L. Singleton, Jr., *Nucl. Phys.* **B518** (1998) 319; R.G. Edwards, U.M. Heller and R. Narayanan, hep-lat/9802016, to appear in *Nucl. Phys. B.*
3. R. Narayanan, these proceedings.
4. R.G. Edwards, U.M. Heller and R. Narayanan, these proceedings.
5. J. Fingberg, U. Heller and F. Karsch, *Nucl. Phys.* **B392** (1993) 493.
6. T. Eguchi, P.B. Gilkey and A.J. Hanson, *Phys. Rep.* **66** (1980) 213.
7. M. Shifman, *Prog. Part. Nucl. Phys.* **39** (1997) 1.
8. E. Witten, *Nucl. Phys.* **B202** (1982) 253.
9. G. 't Hooft, *Comm. Math. Phys.* **81** (1981) 267.
10. E. Cohen and C. Gomez, *Phys. Rev. Lett.* **52** (1984) 237.
11. R.G. Edwards, U.M. Heller and R. Narayanan, hep-lat/9806011, to appear in *Phys. Lett. B.*